

Last time

- Defined semi direct products

Setup: H, K gps, $\phi: K \rightarrow \text{Aut}(H) = \{\text{gp isoms } H \rightarrow H\}$.

$\leadsto H \rtimes_{\phi} K = \{\text{ordered pairs } (h, k) : h \in H, k \in K\}$

w/ multiplication

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1 \phi_{k_1}(h_2), k_1 k_2)$$

Claim: A homom $\phi: K \rightarrow \text{Aut}(H)$ is equivalent to an action of K on H by gp homoms.

Pf: Suppose $\phi: K \rightarrow \text{Aut}(H)$ is given. Then one defines $K \curvearrowright H$ by

$$k \star h = \phi_k(h).$$

This is an action: $k_1 \star (k_2 \star h) = \phi_{k_1}(\phi_{k_2}(h))$
 $= (\phi_{k_1} \circ \phi_{k_2})(h) = \phi_{k_1 k_2}(h) = (k_1 k_2) \star h.$
 $\uparrow$
 b/c ϕ is a gp homom

Action by gp homoms: $k \star (h_1 h_2) = \phi_k(h_1 h_2) = \phi_k(h_1) \phi_k(h_2)$
 $= (k \star h_1) (k \star h_2).$

Conversely, suppose given $K \curvearrowright H$ via gp homom's.

Define: $\phi: K \rightarrow \text{Aut}(H)$ as

$$\phi_k: H \rightarrow H \text{ by } \phi_k(h) = k \cdot h.$$

- By assumption, ϕ_k is a gp homom.
- Because $K \curvearrowright H$ is an action, $\phi_k: H \rightarrow H$ is a bijection
 $\Rightarrow \phi_k \in \text{Aut}(H).$

Also: ϕ is a gp homom.

Pf:
$$\begin{aligned}\phi_{k_1 k_2}(h) &= (k_1 k_2) \cdot h = k_1 \cdot (k_2 \cdot h) \\ &= \phi_{k_1}(\phi_{k_2}(h)) \\ &= (\phi_{k_1} \circ \phi_{k_2})(h)\end{aligned}$$

$$\Rightarrow K \curvearrowright H \rightsquigarrow \phi: K \rightarrow \text{Aut}(H)$$

by gp homom

Thm $H \rtimes_{\phi} K$ is a gp

b) $H \rtimes_{\phi} K \rightarrow K$ is a surj. gp homom w/ kernel H
 $(h, k) \mapsto k$

c) If $\phi: K \rightarrow \text{Aut}(H)$ is the trivial map, i.e. $\phi_k = \text{Id}_H \ \forall k \in K$
then $H \rtimes_{\phi} K \cong H \times K$, the direct product.

Pf: a) Associativity,

$$\begin{aligned}((h_1, k_1)(h_2, k_2))(h_3, k_3) &= (h_1 \phi_{k_1}(h_2), k_1 k_2)(h_3, k_3) \\ &= (h_1 \phi(h_2) \phi(k_1) \phi(h_3), k_1 k_2 k_3)\end{aligned}$$

$$((h_1, k_1)(h_2, k_2)(h_3, k_3)) = ((h_1, k_1)(h_2, k_2)(h_3, k_3))$$

$$= (h_1, \phi_{k_1}(h_2) \phi_{k_1 k_2}(h_3), k_1 k_2 k_3)$$

$$(h_1, k_1)((h_2, k_2)(h_3, k_3)) = (h_1, k_1)(h_2 \phi_{k_2}(h_3), k_2 k_3)$$

$$= (\phi_{k_1}(h_2 \phi_{k_2}(h_3)), k_1 k_2 k_3)$$

Note: $\phi_{k_1}(\phi_{k_2}(h_3)) = \phi_{k_1}(h_2) \phi_{k_1}(\phi_{k_2}(h_3))$

$$= \phi_{k_1}(h_2) \phi_{k_1 k_2}(h_3)$$

identity $(1, 1) : (h, k)(1, 1) = (h \phi_k(1), k) = (h \cdot 1, k) = (h, k)$

$$(1, 1)(h, k) = (1 \cdot \phi_1(h), k) = (1 \cdot h, k) = (h, k)$$

inverses Define $(h, k)' = (\phi_{k^{-1}}(h^{-1}), k^{-1})$

$$(h, k)(h, k)' = (h, k)(\phi_{k^{-1}}(h^{-1}), k^{-1})$$

$$= (h \phi_k(\phi_{k^{-1}}(h^{-1})), k \cdot k^{-1})$$

$$= (h \cdot h^{-1}, k k^{-1}) = (1, 1).$$

$$(h, k)'(h, k) = (\phi_{k^{-1}}(h^{-1}), k^{-1})(h, k)$$

$$= (\phi_{k^{-1}}(h^{-1}) \phi_{k^{-1}}(h), k^{-1} k)$$

$$= (\phi_{k^{-1}}(h^{-1} \cdot h), 1)$$

$$= (\phi_{k^{-1}}(1), 1) = (1, 1)$$

end of part c

b) clear (The map is $(h, k) \mapsto k$)

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c) Clear ~~map~~

Examples

① G gp, $H \leq G$ is normal, $K \leq G$ a subgp, then
 $K \curvearrowright H$ by conjugation $k \cdot h = khk^{-1}$.

① $K = GL_n(\mathbb{R})$, $H = (\mathbb{R}^n, +)$

$\phi: K \rightarrow \text{Aut}(H)$ on

$$\phi_S(v) = S(v)$$

restrict $GL_n(\mathbb{R})$ action on
 $(\mathbb{R}^n, +)$ to O_n action

② $K = O_n$, $H = T_n$

$\phi: O_n \rightarrow \text{Aut}(T_n)$ is

$$\phi_p(t_v) = t_{p(v)}$$

③ $K = GL_a(\mathbb{R}) \times GL_b(\mathbb{R})$, $H = (M_{a,b}(\mathbb{R}), +)$

$\phi: K \rightarrow \text{Aut}(H)$ on

$$\phi_{(g_1, g_2)}(m) = g_1 m g_2^{-1}$$

This is an action by gp homom's because

$$g_1(m+m')g_2^{-1} = g_1 m g_2^{-1} + g_1 m' g_2^{-1}$$

④ $K = (\mathbb{Z}/n\mathbb{Z})^\times$, $H = (\mathbb{Z}/n\mathbb{Z}, +)$

$\phi: K \rightarrow \text{Aut}(H)$ is

$$\psi: K \rightarrow \text{Aut}(H) \quad \psi(k) = \dots$$

$$\phi: K \rightarrow \text{Aut}(H) \quad \text{in}$$

$$\phi_k(h) = k \cdot h$$

Can construct a semi direct product from all the examples above

Ex 2 $T_n \rtimes O_n \cong M_n$ the sp of sym. of $\mathbb{R}^n$

Ex 3 $M_{a,b}(\mathbb{R}) \rtimes (GL_a(\mathbb{R}) \times GL_b(\mathbb{R})) \cong \left\{ \begin{pmatrix} g_1 & x \\ 0 & g_2 \end{pmatrix} : \begin{matrix} g_1 \in GL_a(\mathbb{R}) \\ g_2 \in GL_b(\mathbb{R}) \\ x \in M_{a,b}(\mathbb{R}) \end{matrix} \right\}$

$$\subseteq GL_{a+b}(\mathbb{R})$$

$$h \in M_{a,b}(\mathbb{R}) \longmapsto \begin{pmatrix} I_a & h \\ 0 & I_b \end{pmatrix}$$

$$(g_1, g_2) \in GL_a(\mathbb{R}) \times GL_b(\mathbb{R}) \longmapsto \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

Check: The above defines a gp hom.

Propn Suppose G a gp, $H, K \subseteq G$ are subgps with

- H is normal
- $H \cap K = \{1\}$
- $HK = \{hk : h \in H, k \in K\} = G$

Then $G \cong H \rtimes K \quad \phi: K \rightarrow \text{Aut}(H) \quad \text{in}$

Then $G \cong H \rtimes_{\phi} K$, $\phi: K \rightarrow \text{Aut}(H)$ as
 $\phi_k(h) = khk^{-1}$.

Note: The three assumptions hold for semidirect products.

Pf: First note every elt g of G is uniquely expressible as a product $g = hk$. Indeed, if

$$h_1 k_1 = h_2 k_2, \text{ then}$$

$$\Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K = \{1\}$$

$$\Rightarrow h_2 = h_1, k_2 = k_1$$

Now, define $\phi: G \rightarrow H \rtimes_{\phi} K$ by

$$\phi(hk) = (h, k)$$

It's clear this is a bijection of sets. It is a sp homom. b/c

$$\phi((h_1 k_1)(h_2 k_2)) = \phi(\underline{h_1 k_1 h_2 k_1^{-1}} \underline{k_1 k_2})$$

$$= (h_1 k_1 h_2 k_1^{-1}, k_1 k_2)$$

$$= (\underline{h_1 \phi_{k_1}(h_2)}, \underline{k_1 k_2})$$

$$= (h_1, k_1)(h_2, k_2)$$

$$= \phi(h_1 k_1) \phi(h_2 k_2).$$

□

Example

$$G = D_n, \quad H = C_n = \{1, x, x^2, \dots, x^{n-1}\}.$$

$$K = \{1, s\}$$

$$K = \{1, y\}$$

Note: $H \cap K = \{1\}$, H is normal, $HK = G$.

$$\Rightarrow D_n \cong C_n \rtimes_{\phi} \{1, y\}$$

$$\text{with } \phi_y(x^r) = x^{-r}.$$